Master Modélisation Aléatoire 2ème année
Examen partiel de Processus Stochastiques en Finance
16 novembre 2017, 15h15–18h15, amphi 13E
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Exercise 1 (Discrete hedging in the Black-Scholes model, **12 points**). One of the less realistic assumptions of the Black-Scholes model is the continuous readjustment of the hedging portfolio. The goal of this exercise is to quantify the error which appears when the portfolio is readjusted in discrete time.

We assume that the risky asset follows the simplified martingale Black-Scholes dynamics.

$$dS_t = \sigma S_t dW_t,$$

where W is the standard Brownian motion under the probability $\mathbb{P}$. The risk-free asset evolves with **zero interest rate**: $S_t^0 \equiv 1$.

Consider a European option with time to maturity T and pay-off $g(S_T)$, where g is a continuous function with polynomial growth. Under these assumptions the option price at time t for the underlying value S is given by $C(t,S) = \mathbb{E}[g(Se^{-\frac{\sigma^2}{2}(T-t)+\sigma W_{T-t}})]$, and the optimal hedging strategy is $\delta_t = \frac{\partial C}{\partial S}(t, S_t)$. Recall that the price C satisfies the Black-Scholes equation which in the absence of interest rate takes the form

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0. \tag{1}$$

We assume that the agent readjusts the portfolio at discrete dates $t_i = ih$, $h = \frac{T}{n}$, and applies the discretized hedging strategy $\delta_t^h = \delta_{l^h(t)}$, where $l^h(t) := \sup\{t_i : t_i \leq t\}$ is the closest discretization point to t from the left. We also introduce the notation $r^h(t) := \inf\{t_i : t_i > t\}$ for the closest discretization point to t from the right. The discretization error is then given by

$$\varepsilon_T^h = V_T - V_T^h,$$

where V_T is the terminal value of the portfolio using the continuous-time strategy δ and V_T^h is the terminal value of the portfolio using the discretized strategy δ^h. Our goal is to show that

$$\frac{1}{h}\mathbb{E}\left[(\varepsilon_T^h)^2\right]$$

converges to a finite limit to be identified as $h \to 0$.

First part In this part we shall assume that the price C satisfies

$$\mathbb{E}\left[\int_0^T S_t^4 \left(\frac{\partial^2 C(t,S_t)}{\partial S^2}\right)^2 dt\right] < \infty \tag{2}$$

1. Recall the expressions for V_T and V_T^h and show that the hedging error ε_T^h is represented as a stochastic integral.
2. From the previous question deduce that the L^2 hedging error is given by
$$\mathbb{E}\left[(\varepsilon_T^h)^2\right] = \sigma^2 \int_0^T \mathbb{E}\left[S_t^2(\delta_t - \delta_t^h)^2\right] dt$$
3. Show that by introducing a new probability $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, one can write
$$\mathbb{E}\left[(\varepsilon_T^h)^2\right] = \sigma^2 \int_0^T e^{\sigma^2 t}\mathbb{E}^{\tilde{\mathbb{P}}}\left[(\delta_t - \delta_t^h)^2\right] dt$$

4. Applying the Itô formula to the function $\frac{\partial C}{\partial S}$ and using the Black-Scholes equation (1) show that

$$\delta_t - \delta^h_t = \int_{l^h(t)}^t \frac{\partial^2 C(u,S_u)}{\partial S^2} S_u \sigma d\tilde{W}_u + \int_{l^h(t)}^t \frac{\partial^2 C(u,S_u)}{\partial S^2} S_u \sigma^2 du,$$

where $\tilde{W}$ is a standard Brownian motion under $\tilde{\mathbb{P}}$.

5. From the previous question, using Hölder and Cauchy-Schwarz inequalities, deduce that

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[(\delta_t - \delta^h_t)^2\right] = (1 + O(\sqrt{h}))\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \left(\frac{\partial^2 C(u,S_u)}{\partial S^2}\right)^2 S_u^2 \sigma^2 du\right]$$

6. Coming back to the original probability $\mathbb{P}$, use the integration by parts to show that

$$\mathbb{E}\left[(\varepsilon^h_T)^2\right] = (1 + O(\sqrt{h}))\sigma^4 \mathbb{E}\left[\int_0^T (r^h(t) - t) S_t^4 \left(\frac{\partial^2 C(t,S_t)}{\partial S^2}\right)^2 dt\right]$$

7. Let f be a Borel function such that $\int_0^T |f(t)|dt < \infty$. Show that

$$\lim_{h\to 0} \frac{1}{h}\int_0^T (r^h(t) - t) f(t) dt = \frac{1}{2}\int_0^T f(t)dt.$$

Hint: use an approximation of f by an increasing sequence of piece-wise constant functions.

8. Use the previous result to compute the limit of

$$\frac{1}{h}\mathbb{E}\left[(\varepsilon^h_T)^2\right].$$

Second part In this part we will show that the property (2) is satisfied in the case of call and put options.

9. Show that if $g(x) = (x - K)^+$ or $g(x) = (K - x)^+$ then

$$\mathbb{E}\left[\int_0^T S_t^4 \left(\frac{\partial^2 C(t,S_t)}{\partial S^2}\right)^2 dt\right] = \frac{K^2}{\sigma^2}\int_0^T \frac{\mathbb{E}\left[n^2(d_t)\right]}{T-t} dt,$$

where n is the standard normal density and $d_t = \frac{\log \frac{S_t}{K} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$.

10. From the previous question deduce that the expectation of (2) is finite.

Exercise 2 (Implied volatility in the shifted log-normal model, **8 points**).

First part We consider the following stochastic differential equation

$$dS_t = (S_t - \delta)\sigma dW_t, \quad S_0 > \delta, \tag{3}$$

where $\delta > 0$ and W is a standard Brownian motion.

1. Using a change of variable, write down the explicit solution for (3).
2. Write down explicit formulas for the price of European call and put options on S with strike $K > \delta$. What is the price of a put option on S with strike $K \le \delta$?
3. Give the expression of the local volatility in the model (3).

4. Recall the definition of the implied volatility $I(T, K)$ in a given arbitrage-free model. Compute the limiting implied volatility as $T \to 0$ in the model (3) through the asymptotic formula of the course:

$$\tilde{I}(0, x) = \left\{ \int_0^1 \frac{dy}{\tilde{\sigma}(0, xy)} \right\}^{-1},$$

where $\tilde{I}$ and $\tilde{\sigma}$ are, respectively, the implied and local volatility expressed in terms of the moneyness variable $x = \log \frac{S_0}{K}$.

Second part The goal of this part is to compute the limiting implied volatility directly.

5. Using the asymptotic equivalent $N(x) \sim \frac{n(x)}{|x|}$, $x \to -\infty$, where $N(x)$ is the standard normal distribution function and $n(x)$ is the standard normal density, determine the asymptotic equivalent as $T \to 0$ for the Black-Scholes price of an out-of-the money call option with volatility σ. This price will be denoted by $C^{BS}(S_0, T, K, \sigma)$.

6. Consider a general arbitrage-free model where the call option price is denoted by $C(S_0, T, K)$. The implied volatility in this model will be denoted by $I(T, K)$.

   (a) What can be said about the behavior of $TI^2(T, K)$ as $T \to 0$?

   (b) Using the previous property and the definition of the implied volatility, show that an equivalent similar to the one of question 5 may be obtained for the price of out-of-the money call option price $C(S_0, T, K)$ as $T \to 0$, in terms of the implied volatility $I(T, K)$.

7. From the previous question, deduce that the implied volatility in any arbitrage-free model admits the following asymptotic equivalent

$$I^2(T, K) \sim -\frac{\log^2 \frac{S_0}{K}}{2T \log C(S_0, T, K)}, \quad T \to 0, \quad K > S_0.$$

8. From questions 2, 5 and 7, deduce the limiting implied volatility in the model (3) as $T \to 0$.

Correction

Exo 1

Première partie

1. L'erreur de couverture vérifie

$$\varepsilon_T^h = V_T - V_T^h = \int_0^T (\delta_t - \delta_t^h) dS_t = \int_0^T (\delta_t - \delta_t^h)\sigma S_t dW_t.$$

2. Par l'isométrie d'Itô et le théorème de Fubini,

$$\mathbb{E}\left[(\varepsilon_T^h)^2\right] = \sigma^2 \int_0^T \mathbb{E}\left[S_t^2(\delta_t - \delta_t^h)^2\right] dt$$

3. On définit la probabilité $\tilde{\mathbb{P}}$ via

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{S_T^2}{\mathbb{E}[S_T^2]} = e^{-\sigma^2 T} S_T^2.$$

4. Rappelons que $\tilde{W}_t := W_t - 2\sigma t$ est un mouvement Brownien standard sous $\tilde{\mathbb{P}}$. Une application de la formule d'Itô à $\delta_t - \delta_t^h = \frac{\partial C}{\partial s}(t, S_t) - \frac{\partial C}{\partial s}(l(t), S_{l(t)})$ donne

$$\delta_t - \delta_t^h = \int_{l(t)}^t \frac{\partial^2 C}{\partial S^2} S_t \sigma \tilde{W}_t + 2\int_{l(t)}^t \frac{\partial^2 C}{\partial S^2} S_t \sigma^2 dt + \int_{l(t)}^t \frac{\partial^2 C}{\partial S^2} S_t r dt + \int_{l(t)}^t \frac{\partial^2 C}{\partial S \partial t} dt + \frac{1}{2}\int_{l(t)}^t \frac{\partial^3 C}{\partial S^3} S_t^2 \sigma^2 dt.$$

D'un autre coté, en dérivant chaque terme dans l'équation de Black-Scholes par rapport à S, on a

$$\frac{\partial^2 C}{\partial S \partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^3 C}{\partial S^3} + \sigma^2 S \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial^2 C}{\partial S^2} = 0.$$

En substituant cette expression dans celle au-dessus, nous obtenons le résultat voulu.

5. Par isométrie d'Itô, on a

$$\mathbb{E}^{\tilde{\mathbb{P}}}[(\delta_t - \delta_t^h)^2] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \left(\frac{\partial^2 C}{\partial S^2}\right)^2 S_u^2 \sigma^2 du\right] + \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma^2 du\right)^2\right] + 2\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma dW_u \times \int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma^2 du\right].$$

Pour estimer le deuxième terme, on utilise l'inégalité de Hölder (pour l'intégrale):

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma^2 du\right)^2\right] \le (t - l^h(t))\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \left(\frac{\partial^2 C}{\partial S^2}\right)^2 S_u^2 \sigma^4 du\right].$$

Pour le troisième terme, on applique l'inégalité de Cauchy-Schwartz (à l'espérance):

$$\left|\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma dW_u \times \int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma^2 du\right]\right|$$

$$\leq \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma dW_u\right)^2\right]^{1/2} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\int_{l^h(t)}^t \frac{\partial^2 C}{\partial S^2} S_u \sigma^2 du\right)^2\right]^{1/2}$$

$$\leq \sigma\sqrt{t - l^h(t)}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \left(\frac{\partial^2 C}{\partial S^2}\right)^2 S_u^2 \sigma^2 du\right].$$

6. Une intégration par parties donne

$$\int_0^T dt e^{\sigma^2 t}\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_{l^h(t)}^t \left(\frac{\partial^2 C(u, S_u)}{\partial S^2}\right)^2 S_u^2 \sigma^2 du\right]$$

$$= \int_0^T dt \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\frac{\partial^2 C(t, S_t)}{\partial S^2}\right)^2 S_t^2 \sigma^2\right] \int_t^{r^h(t)} e^{\sigma^2 u} du$$

$$= (1 + O(h)) \int_0^T dt (r^h(t) - t) e^{\sigma^2 t} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\left(\frac{\partial^2 C(t, S_t)}{\partial S^2}\right)^2 S_t^2 \sigma^2\right]$$

$$= (1 + O(h)) \int_0^T dt (r^h(t) - t) \mathbb{E}\left[\tilde{S}_t^2 \left(\frac{\partial^2 C(t, S_t)}{\partial S^2}\right)^2 S_t^2 \sigma^2\right]$$

7. Soit $g_h(t) = \frac{r^h(t)-t}{h}$. Pour une fonction u constante par morceaux on a clairement

$$\lim_{h\downarrow 0} \int_0^T g_h(t) u(t) dt = \frac{1}{2}\int_0^T u(t) dt.$$

Soit $(f_n)_{n\geq 1}$ une suite de fonctions constantes par morceaux telle que $f_n(t) \leq f_{n+1}(t) \leq f(t)$ et $\lim_{n\to\infty} f_n(t) = f(t)$ pour tout $t \in [0, T]$. Alors par convergence monotone, puisque $|g_h| \leq 1$,

$$\lim_{n\to\infty} \int_0^T (f(t) - f_n(t)) g_h(t) dt = 0$$

uniformément en h, ce qui montre que

$$\lim_{h\downarrow 0} \int_0^T f(t) g_h(t) dt = \frac{1}{2}\int_0^T f(t) dt.$$

8. Par le résultat précédent, on a immédiatement

$$\lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}[(\varepsilon_T^h)^2] = \frac{\sigma^4}{2}\mathbb{E}\left[\int_0^T S_t^4 \left(\frac{\partial^2 C(t, S_t)}{\partial S^2}\right)^2 dt\right]$$

et donc

$$\lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}[(\varepsilon_T^h)^2] = \frac{\sigma^4}{2}\mathbb{E}\left[\int_0^T S_t^4 \left(\frac{\partial^2 C(t, S_t)}{\partial S^2}\right)^2 dt\right].$$

Deuxième partie

1. Par le résultat du cours nous avons

$$\frac{\partial^2 C}{\partial S^2} = \frac{1}{S\sigma\sqrt{T-t}} n(d_+) = \frac{K}{S\sigma\sqrt{T-t}} n(d_-),$$

   où la définition de d_- coïncide avec le coefficient d_t du problème.

2. Un calcul direct montre

$$\mathbb{E}[n^2(d_t)] = \sqrt{\frac{T-t}{T+t}} e^{-\frac{C^2}{T+t}} \leq \sqrt{\frac{T-t}{T+t}}, \quad \text{où} \quad c = \frac{1}{\sigma}\log\frac{S_0}{K} - \frac{\sigma T}{2}.$$

   Le fait que l'intégrale en question est finie en découle immédiatement.

Exo 2

1.

$$S_t = \delta + (S_0 - \delta) e^{\sigma W_t - \frac{\sigma^2}{2} t}.$$

2.

$$C(S_0, T, K) = \mathbb{E}\left[\left(\delta + (S_0 - \delta) e^{\sigma W_t - \frac{\sigma^2}{2} t} - K\right)^+\right] = C^{BS}(S_0 - \delta, T, K - \delta, \sigma),$$

   and similarly for the put. For $K < \delta$ the put has zero price.

3. $\sigma(t, S) = \frac{S-\delta}{S}$.

4. The implied volatility satisfies $C(S_0, T, K) = C^{BS}(S_0, T, K, \sigma)$. We have $\tilde{\sigma}(x) = 1 - \frac{\delta}{S_0} e^x$, so that

$$\int_0^1 \frac{dy}{\tilde{\sigma}(0, xy)} = \int_0^1 \frac{dy}{1 - \frac{\delta}{S_0} e^{xy}} = \frac{\log \frac{S_0 - \delta}{K - \delta}}{\log S_0/K}.$$

5. Substituting the asymptotic equivalent for N, and neglecting lower-order terms, we have

$$C^{BS}(S_0, T, K, \sigma) \sim \frac{S n(d_1) \sigma^3 T^{3/2}}{\log^2(S_0/K)}$$

6. Absence of arbitrage implies that $I^2(T, K)T \to 0$ as $T \to 0$. Similarly to the previous question, we then get

$$C(S_0, T, K) = C^{BS}(S_0, T, K, I(T, K)) \sim \frac{S n(d_1(I)) I^3(T, K) T^{3/2}}{\log^2(S_0/K)},$$

   where $d_1(I)$ is computed with $I(T, K)$ instead of σ.

7. By the previous question,

$$\lim_{T \to 0} \frac{S n(d_1(I)) I^3(T, K) T^{3/2}}{C(S_0, T, K) \log^2(S_0/K)} = 1.$$

   Taking the logarithm,

$$\lim_{T \to 0} \left\{ -\frac{d_1^2(I)}{2} - \log C(S_0, T, K) + \log(I^3(T, K) T^{3/2}) \right\} = c,$$

where c is a constant. Expanding $d_1^2(I)$, multiplying by $I^2(T,K)T$ and omitting negligible terms, we then get

$$\lim_{T\to 0}\left\{\frac{\log^2(S_0/K)}{2} + I^2(T,K)T\log C(S_0,T,K)\right\} = 0,$$

which finishes the proof.

8.

$$I^2(T,K) \sim -\frac{\log^2\frac{S_0}{K}}{2T\log C^{BS}(S_0-\delta,T,K-\delta,\sigma)} \sim -\frac{\log^2\frac{S_0}{K}}{2T\log C(S_0,T,K)} \sim \sigma^2\frac{\log^2\frac{S_0}{K}}{\log^2\frac{S_0-\delta}{K-\delta}}$$